Applied Synthetic Computability Theory

Gödel's Incompleteness and Post's Problem

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LIPN, February 27th, 2025

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$$\chi_K(M) := \begin{cases} 1 & \text{if } M \text{ terminates} \\ 0 & \text{if } M \text{ diverges} \end{cases}$$

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- Totality: given M, since M either terminates or diverges by the law of excluded middle, we have either $\chi_K(M) = 1$ or $\chi_K(M) = 0$, respectively

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So classical logic is needed to show that $\chi_K(M)$ really is a function!

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In any of those: no need for Turing machines, simply treat all functions as computable

Some Synthetic Definitions[1]

$P \subseteq X$ is decidable if there exists $d : X \to \mathbb{B}$ with $x \in P \leftrightarrow d\,x = \mathsf{true}$

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$P \subseteq X$ is semi-decidable if there exists $s : X \times \mathbb{N} \to \mathbb{B}$ with $x \in P \leftrightarrow \exists n.\, s\,(x, n) = \mathsf{true}$

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$P \subseteq X$ reduces to $Q \subseteq Y$ if there exists $r : X \to Y$ with $x \in P \leftrightarrow r\,x \in Q$

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$$e\,n := \begin{cases} n_2 & n = \langle n_1, n_2 \rangle \text{ and } s(n_1, n_2) = \text{true} \\ * & \text{otherwise} \end{cases}$$

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   $\Rightarrow$ Given $d : Y \to \mathbb{B}$ and $f : X \to Y$ pick $d \circ f$

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5. Conclude that d decides P. □

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4. Assume an enumeration of the partial function space $\mathbb{N} \rightharpoonup \mathbb{N}$

EPF and the Halting Problem

Axiom (EPF, Richman (1983); Bauer (2006); Forster (2022))

There is a universal function $\Theta : \mathbb{N} \to (\mathbb{N} \rightharpoonup \mathbb{N})$ *enumerating all partial functions:*

$$\forall f : \mathbb{N} \rightharpoonup \mathbb{N}. \exists c : \mathbb{N}. \forall xb. \Theta_c\, x \downarrow b \leftrightarrow f\, x \downarrow b$$

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Proof.

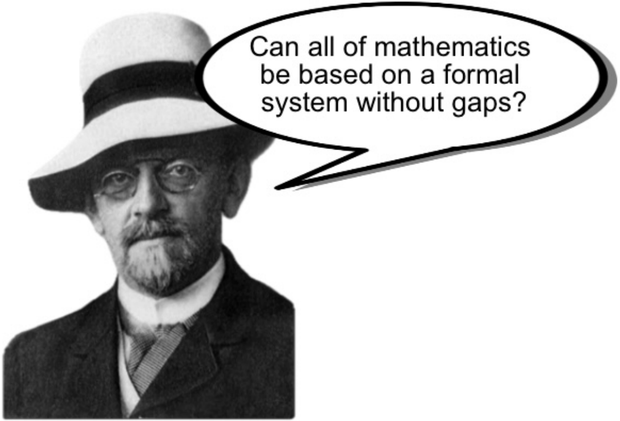
Assume $d : \mathbb{N} \to \mathbb{B}$ decides K. Consider the function $f : \mathbb{N} \rightharpoonup \mathbb{B}$ with $f\, c \uparrow$ if $d\, c = \mathsf{true}$ and $f\, c \downarrow \mathsf{true}$ otherwise. Let c be the code of f given by EPF, we derive a contradiction:

$$d\, c = \mathsf{true} \;\Leftrightarrow\; c \in K \;\Leftrightarrow\; \Theta_c\, c \downarrow \Leftrightarrow\; f\, c \downarrow \Leftrightarrow\; f\, c \downarrow \mathsf{true} \;\Leftrightarrow\; d\, c = \mathsf{false} \qquad \square$$

Application 1: Gödel's Incompleteness

(jww. Marc Hermes and Benjamin Peters)

Historical Motivation



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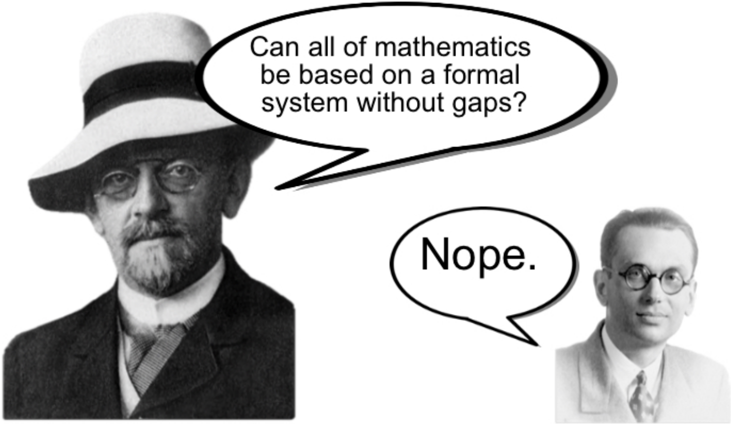

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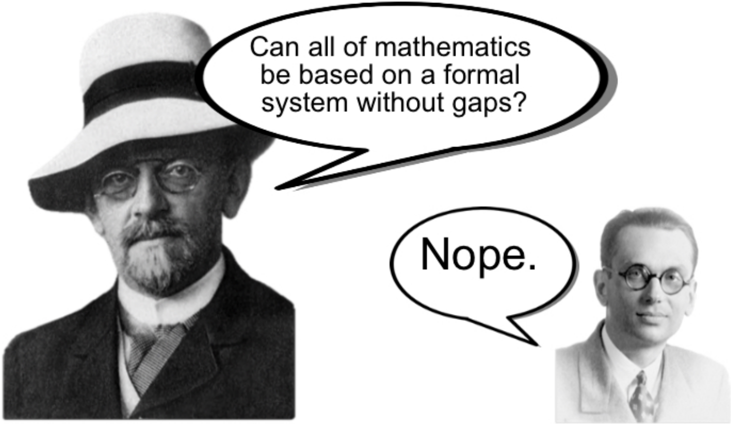

"Every sufficiently strong formal system admits independent sentences."

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- Turing(/Post): Gödel's incompleteness follows from undecidability (Turing, 1937)

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- Turing(/Post): Gödel's incompleteness follows from undecidability (Turing, 1937)
- Kleene: Rosser's incompleteness follows from recursive inseparability (Kleene, 1951)

Matrix of Incompleteness Theorems

	Disprove completeness	Independent sentence
Soundness	Turing	Gödel
ω-consistency		Gödel
Consistency		Rosser/Kleene

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  Scott Aaronson on his blog
- *"Here I shall present very simple computability-based proofs of Gödel/Rosser's incompleteness theorem, which require only basic knowledge about programs. I feel that these proofs are little known despite giving a very general form of the incompleteness theorems, and also easy to make rigorous without even depending on much background knowledge in logic."*
  User21820 on StackExchange

Motivational Testimonies (ctd.)

Gödel's incompleteness theorems [edit]

The concepts raised by Gödel's incompleteness theorems are very similar to those raised by the halting problem, and the proofs are quite similar. In fact, a weaker form of the First Incompleteness Theorem is an easy consequence of the undecidability of the halting problem. This weaker form differs from the standard statement of the incompleteness theorem by asserting that an axiomatization of the natural numbers that is both complete and sound is impossible. The "sound" part is the weakening: it means that we require the axiomatic system in question to prove only *true* statements about natural numbers. Since soundness implies consistency, this weaker form can be seen as a corollary of the strong form. It is important to observe that the statement of the standard form of Gödel's First Incompleteness Theorem is completely unconcerned with the truth value of a statement, but only concerns the issue of whether it is possible to find it through a mathematical proof.

`https://en.wikipedia.org/wiki/Halting_problem`

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Definition

A formal system $\mathcal{S} = (\mathbb{S}, \vdash, \neg)$ consists of:

- $\mathbb{S}$ is a set we consider the collection of formal sentences
- $\vdash$ is a semi-decidable subset we consider the provable sentences
- $\neg : \mathbb{S} \to \mathbb{S}$ is a computable function we consider negation, satisfying consistency:

$$\forall \varphi \in \mathbb{S}.\ \neg(\vdash \varphi \land \ \vdash \neg\varphi)$$

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Lemma (Refutation)

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Proof.

That $\nvdash \varphi$ implies $\vdash \neg\varphi$ is by completeness, the other direction by consistency. $\square$

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Theorem

Let P be some undecidable set, for instance the halting problem K*. If P reduces to the provable sentences $\vdash$ of a formal system $\mathcal{S} = (\mathbb{S}, \vdash, \neg)$, then $\mathcal{S}$ cannot be complete.*

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4. Derive that $\vdash$ is decidable by (Post).
5. Conclude that P must be decidable by (Reduction).

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Let P be some undecidable set, for instance the halting problem K*. If P reduces to the provable sentences $\vdash$ of a formal system $\mathcal{S} = (\mathbb{S}, \vdash, \neg)$, then $\mathcal{S}$ cannot be complete.*

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1. Assume that P is undecidable and that $\mathcal{S}$ were complete.
2. Obtain that that $\nvdash \varphi$ iff $\vdash \neg\varphi$ by (Refutation).
3. Observe that the complement of $\vdash$ is semi-decidable.
4. Derive that $\vdash$ is decidable by (Post).
5. Conclude that P must be decidable by (Reduction).
6. Contradiction. $\square$

Matrix of Incompleteness Theorems

	Disprove completeness	Independent sentence
Soundness	Turing ($\checkmark$)	Gödel
ω-consistency		Gödel
Consistency		Rosser/Kleene

Halting Problem (Refined)

Lemma

For every partial decider $d : \mathbb{N} \rightharpoonup \mathbb{B}$ *for* $\mathsf{K} = \{c \in \mathbb{N} \mid \Theta_c\, c \downarrow\}$ *with*

$$\forall x.\, x \in \mathsf{K} \;\leftrightarrow\; d\, x \downarrow \mathsf{true}$$

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We first define a partial function $f : \mathbb{N} \rightharpoonup \mathbb{B}$ diagonalising against d by:

$$f\, x := \begin{cases} \mathsf{true} & \text{if } d\, x \downarrow \mathsf{false} \\ \uparrow & \text{otherwise} \end{cases}$$

Now using EPF we obtain a code c for f and deduce that $d\, c \uparrow$ by:

$$d\, c \downarrow \mathsf{true} \;\Leftrightarrow\; c \in \mathsf{K} \;\Leftrightarrow\; \Theta_c\, c \downarrow \Leftrightarrow\; f\, c \downarrow \Leftrightarrow\; f\, c \downarrow \mathsf{true} \;\Leftrightarrow\; d\, c \downarrow \mathsf{false} \qquad \square$$

Post's Theorem (Refined)

Theorem

Given disjoint semi-decidable sets $P, Q \subseteq X$, there is a partial decider $d : X \rightharpoonup \mathbb{B}$ with:

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Proof.

Given s_1 semi-deciding P and s_2 semi-deciding Q, define d by:

$$d\,x\,n := \begin{cases} \mathsf{true} & \text{if } s_1\,x\,n \\ \mathsf{false} & \text{if } s_2\,x\,n \\ \uparrow & \text{otherwise} \end{cases}$$

Then use disjointness to verify well-definedness and specification. $\square$

Partial Deciders of Formal Systems

Since formal systems have two canonical disjoint semi-decidable sets:

Lemma

For every formal system $\mathcal{S} = (\mathbb{S}, \neg, \vdash)$ there is a partial function $d_{\mathcal{S}} : \mathbb{S} \rightharpoonup \mathbb{B}$ with:

$$\forall \varphi.\, (\vdash \varphi \leftrightarrow d_{\mathcal{S}}\, \varphi \downarrow \mathsf{true}) \land (\vdash \neg\varphi \leftrightarrow d_{\mathcal{S}}\, \varphi \downarrow \mathsf{false})$$

Moreover, because of consistency we have $d_{\mathcal{S}}\, \varphi \uparrow$ exactly if φ is an independent sentence.

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- If $\mathcal{S}$ is complete, then $d_{\mathcal{S}}$ induces a decider for representable problems
- Even without completeness, $d_{\mathcal{S}}$ is a partial decider for representable problems...

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If $r : \mathbb{N} \to \mathbb{S}$ reduces K to $\mathcal{S}$, then $d_{\mathcal{S}} \circ r$ is a candidate decider for K. Thus there is some code c with $d_{\mathcal{S}}\,(r\,c)\uparrow$, so $r\,c$ is must be independent. □

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Implicitly, the formal system is assumed to be sound due to the reducibility property:

$$\vdash r(x) \;\to\; x \in \mathsf{K}$$

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- Recursive Inseparability: disjoint sets P, Q that are not separable by $d : X \to \mathbb{B}$

$$\forall x.\, (x \in P \to d\, x = \mathsf{true}) \land (x \in Q \to d\, x = \mathsf{false})$$

Canonical Inseparable Sets

Lemma

The sets $\mathsf{K}^1 := \{c \in \mathbb{N} \mid \Theta_c\, c \downarrow \mathsf{true}\}$ *and* $\mathsf{K}^0 := \{c \in \mathbb{N} \mid \Theta_c\, c \downarrow \mathsf{false}\}$ *are semi-decidable but recursively inseparable, in fact for every partial separation* $d : \mathbb{N} \rightharpoonup \mathbb{B}$ *with*

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We say that a formal system $\mathcal{S}$ separates sets $P, Q \subseteq X$ if there is a function $r : X \to \mathbb{S}$ with

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Corollary

If $\mathcal{S}$ separates K^1 and K^0, then every extension $\mathcal{S}' \supseteq \mathcal{S}$ has an independent sentence.

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Essential Incompleteness of Robinson's Q

To instantiate these abstract proofs to Q, we need a stronger assumption than EPF:

Axiom ($\mathsf{CT_Q}$)

For every $f : \mathbb{N} \rightharpoonup \mathbb{B}$ there is a Σ_1-formula φ with: $f\,x \downarrow b \;\leftrightarrow\; \mathsf{Q} \vdash \forall b'.\, \varphi(\overline{x}, b') \leftrightarrow b' = \overline{b}$

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CT_Q implies that Q and every consistent extension of it has an independent sentence:

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- Sanity check: $\mathsf{CT_Q}$ is equivalent to CT

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Circumvent CT:

- Work against abstract notion of computable functions
- Instantiate trivially with CT for the constructively minded
- Instantiate with Turing-computability for the classically minded

Application 2: Post's Problem

(jww. Yannick Forster, Niklas Mück, Takako Nemoto, Haoyi Zeng)

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- Synthetic proof mechanised in Coq by Zeng et al. (2024a), relying on Σ_2-LEM
- Analytic proof given by Nemoto (2024), relying only on Σ_1-LEM aka LPO
- Combination yields a synthetic and mechanised proof using LPO (Zeng et al., 2024b)

Synthetic Oracle Computability

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Definition (Forster, Kirst and Mück (2023))

An oracle computation is a functional $F\colon (Q \to A \to \mathbb{P}) \to I \to O \to \mathbb{P}$ captured by a computation tree $\tau\colon I \to A^* \rightharpoonup Q + O$ and its induced interrogation relation $\tau i ; R \vdash qs ; as$ as follows:

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$\mathcal{S}_Q(P)\ :=$ there is an oracle computation $F\colon(\mathbb{N}\to\mathbb{B}\to\mathbb{P})\to\mathbb{N}\to\mathbb{1}\to\mathbb{P}$ with $\mathrm{dom}(F\,Q)=P$

Continuity of Oracle Computations

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Our employed notion of sequential continuity is strictly stronger than modulus-continuity:

Lemma (Forster, Kirst and Mück (2023))

1. *Every oracle computation F is modulus-continuous:*

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2. *Not every modulus-continuous functional is an oracle computation.*

Proof.

1. From a terminating run $F\,R\,i\,o$ we obtain an interrogation $\tau\,i; R \vdash qs; as$ and can easily show that qs is a modulus of continuity.
2. The modulus-continuous functional $F\,R\,i\,o \;:=\; \exists q.\, R\,q\,\text{true}$ is not an oracle computation as for any computation tree τ we can define a suitably blocking oracle. □

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1. Φ_e *is an oracle computation for all* $e : \mathbb{N}$
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3. *The Turing jump* $P'\, x := \Phi_x^P(x) \downarrow \mathrm{true}$ *of* P *is strictly harder than* P

Low Simple Sets and Limit Computability

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$P : X \to \mathbb{P}$ is low if $P' \preceq_T H$ and simple if it is co-infinite, semi-decidable, and for W_e being the e-th enumerable set we have $W_e \cap P \neq \emptyset$ whenever W_e is infinite.

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Definition (Shoenfield (1959) and Gold (1965))

$P : X \to \mathbb{P}$ is limit-computable if there exists a function $f : X \to \mathbb{N} \to \mathbb{B}$ with

$$x \in P \;\leftrightarrow\; \exists n.\forall m > n.\ f(x, m) = \text{true} \quad \wedge \quad x \notin P \;\leftrightarrow\; \exists n.\forall m > n.\ f(x, m) = \text{false}.$$

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$\Rightarrow$ Every low simple set is a solution to Post's problem!

Definition (Shoenfield (1959) and Gold (1965))

$P : X \to \mathbb{P}$ is limit-computable if there exists a function $f : X \to \mathbb{N} \to \mathbb{B}$ with

$$x \in P \;\leftrightarrow\; \exists n.\forall m > n.\ f(x, m) = \text{true} \quad \wedge \quad x \notin P \;\leftrightarrow\; \exists n.\forall m > n.\ f(x, m) = \text{false}.$$

$\Rightarrow$ Limit-computability provides easy way to prove lowness...

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Lemma (1)

If $\mathcal{S}_Q(P)$ *and* $\mathcal{S}_Q(\overline{P})$ *then* $P \preceq_T Q$.

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Proof.

If P is limit computable, then immediately by definition both P and $\overline{P}$ are Σ_2. Moreover, since the halting problem H is Σ_1, Lemma 2 together with LPO yields both $\mathcal{S}_H(P)$ and $\mathcal{S}_H(\overline{P})$. From there we conclude $P \preceq_T H$ with Lemma 1. □

The Priority Method

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Fix step function $\gamma : \mathbb{N}^* \to \mathbb{N} \to \mathbb{N} \to \mathbb{P}$, approximate S inductively:

$$\frac{}{0 \leadsto [\,]} \qquad \frac{n \leadsto L \quad \gamma_n^L\ x}{n+1 \leadsto x :: L} \qquad \frac{n \leadsto L \quad \forall x.\ \neg\ \gamma_n^L\ x}{n+1 \leadsto L}$$

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- S satisfies $P_e :=$ W_e is infinite $\to W_e \cap S \neq \emptyset$ $\Rightarrow$ S is simple
- S satisfies $N_e := (\exists^\infty n.\ \Phi_e^S(e)[n]\downarrow) \to \Phi_e^S(e)\downarrow$ $\Rightarrow$ S' is limit computable (using LPO)

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Define suitable γ again relative to a wall function ω of same type:

- $\omega_n^L(e) \geq 2 \cdot e \;\Rightarrow\; S$ satisfies the requirements P_e
- $\omega_n^L(e) \geq \max_{e' \leq e} U_{e'}^L(e')[n] \;\Rightarrow\; S$ satisfies the requirements N_e (using LPO)

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Theorem

Assuming LPO, a low simple set exists.

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Theorem

Assuming LPO, a low simple set exists.

Proof.

Choose the wall function $\omega := \max(2 \cdot e, \max_{e' \leq e} U_{e'}^L(e')[n])$. □

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Reverse analysis:

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Generalisation:

- Friedberg-Muchnik theorem
- Low basis theorem
- Connections to true second-order arithmetic

Conclusion

Synthetic Computability Propaganda

Reviewer 1: “Clearly, synthetic computability is trivializing things that should have been trivial from the beginning.”

1. Guides towards the computational essence of proofs
2. Allows concise but precise formalisation
3. Makes mechanisation feasible

What else could we discuss?

- Constructive status of completeness theorems
  - Model existence is equivalent to WLEM
  - Quasi-completeness is equivalent to WLEMS
- Constructive status of Löwenheim-Skolem theorems
  - Downwards part needs "DC-CC" and "LEM-MP"
  - Upwards part is usually proved via compactness
- Realisability models of constructive type theory and HOL
  - Type theories with choice sequences to separate formulations of MP
  - Effectful realisability interpretation of HOL to unify realisability variants
- Completeness theorems for bi-intuitionistic logic
  - Semantics in (constant-domain) Kripke models and (complete) bi-Heyting algebras

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