Synthetic Versions of the Kleene-Post and Post's Theorem

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Abstract

We discuss our ongoing formalisation of the Kleene-Post theorem ([6], establishing incomparable Turing degrees) and Post's theorem ([9], connecting the arithmetic hierarchy with Turing degrees) using synthetic computability theory in constructive type theory.

Synthetic Oracle Machines We briefly outline the synthetic rendering of oracle machines as described in a related abstract [4], adjusting [2], based on a similar proposal by Bauer [1]. The main idea is that an oracle machine R can be represented as a function operating on functional relations $A : \mathbb{N} \rightsquigarrow \mathbb{B}$ relating (some) natural numbers $n : \mathbb{N}$ to (unique) boolean values $b : \mathbb{B}$:

$$R : (\mathbb{N} \rightsquigarrow \mathbb{B}) \to \mathbb{N} \rightsquigarrow \mathbb{B}$$

The input relation acts as oracle that can be accessed to describe the returned relation. To ensure that this description is effective, we require R to return computable output for computable input, captured as partial functions $f : \mathbb{N} \rightharpoonup \mathbb{B}$, by imposing a computational core

$$r : (\mathbb{N} \rightharpoonup \mathbb{B}) \to \mathbb{N} \rightharpoonup \mathbb{B} \qquad \text{with} \qquad \forall f.\, R\, f = r\, f.$$

Note that here and in the remainder of this text we freely identify partial functions $f : \mathbb{N} \rightharpoonup \mathbb{B}$ with their (functional) graphs $\lambda nb.\, fn = b$, reusing the equality symbol for evaluation of f. To further rule out exotic behaviour, we require R to be continuous in the following sense:

$$\forall (A : \mathbb{N} \rightsquigarrow \mathbb{B}).\forall (n : \mathbb{N}).\forall (b : \mathbb{B}).\ R\, A\, n\, b \to \exists (L : \mathbb{N}^*).\, L \subseteq \mathsf{dom}(A) \land \forall A'.\, A' =_L A \to R\, A'\, n\, b$$

Continuity in this sense expresses that from any terminating run $R\,A\,n\,b$ one can extract a list L of queries to which the oracle A replied, such that $R\,A'\,n$ terminates for all oracles A' agreeing with A on L with the same value b. Observed externally, by all these restrictions and according to the classical, syntactic definition we narrow down the amount of oracle machines to countable extent. In fact, we make this limitation available internally by assuming an enumeration r_n of all computational cores. We currently investigate for which formulations a variant of Church's thesis [7, 10, 11, 3] is enough to obtain such an enumerator.

Given $A, B : \mathbb{N} \to \mathbb{P}$, we call R a Turing reduction from A to B if $R\,B = A$ (reinterpreting predicates as functional relations) and write $A \preceq_T B$ if a Turing reduction from A to B exists. We assume extensionality of functions and relations.

Kleene-Post Theorem To establish incomparable Turing degrees, we adapt the proof given in Odifreddi's textbook [8] to our synthetic setting. The usual strategy is to obtain them as the unions $A := \bigcup_{n:\mathbb{N}} \sigma_n$ and $B := \bigcup_{n:\mathbb{N}} \tau_n$ of cumulative increasing sequences σ_n and τ_n of boolean strings such that the former take care that no r_n induces a reduction $B \preceq_T A$ and the latter conversely rule out $A \preceq_T B$. Naturally, in our synthetic setting we are not able to define these sequences as computable functions $\mathbb{N} \to \mathbb{B}^*$, as this would force A and B decidable. Instead, we characterise both sequences simultaneously with an inductive predicate $\triangleright : \mathbb{N} \to \mathbb{B}^* \to \mathbb{B}^* \to \mathbb{P}$ such that $n \triangleright (\sigma, \tau)$ represents σ_n as σ and τ_n as τ, by adding to $0 \triangleright (\epsilon, \epsilon)$ the rules:

$$\frac{2n \triangleright (\sigma,\tau) \quad \sigma' \geq \sigma \quad b = r_n\, \sigma'\, |\tau|}{2n+1 \triangleright (\sigma', \tau +\!\!+ [\neg b])}\text{E1} \qquad \frac{2n \triangleright (\sigma,\tau) \quad \neg(\exists \sigma' b.\, \sigma' \geq \sigma \land b = r_n\, \sigma'\, |\tau|)}{2n+1 \triangleright (\sigma, \tau +\!\!+ [0])}\text{E2}$$

$$\frac{2n+1 \triangleright (\sigma,\tau) \quad \tau' \geq \tau \quad b = r_n\, \tau'\, |\sigma|}{2n+2 \triangleright (\sigma +\!\!+ [\neg b], \tau')}\text{O1} \qquad \frac{2n+1 \triangleright (\sigma,\tau) \quad \neg(\exists \tau' b.\, \tau' \geq \tau \land b = r_n\, \tau'\, |\sigma|)}{2n+2 \triangleright (\sigma +\!\!+ [0], \tau)}\text{O2}$$

In every even step with $2n \triangleright (\sigma, \tau)$ the sequences are extended such that r_n applied to any prefix of A differs from any prefix of B at position $|\tau|$, either by flipping the result if r_n already converges on some extension $\sigma' \geq \sigma$ (E1) or by setting a dummy value if r_n diverges on all extensions (E2). Dually, in every odd step with $2n + 1 \triangleright (\sigma', \tau')$ it is taken care that r_e applied to any prefix of B differs from any prefix of A.

We state the central lemma used to show $B \not\preceq_T A$, a dual version yields $A \not\preceq_T B$.

Lemma 1. *Let R be an oracle machine factoring through the computational core r_n. If further given $2n \triangleright (\sigma, \tau)$ and $2n + 1 \triangleright (\sigma', \tau')$, then $B\,|\tau|\,b$ implies $\neg(R\,A\,|\tau|\,b)$.*

Theorem 1 (Kleene-Post)**.** *There are predicates A, B such that neither $A \preceq_T B$ nor $B \preceq_T A$.*

Proof. Suppose that $B \preceq_T A$, so $R\,A = B$ for some oracle machine R with core r_n. Given that we try to derive a contradiction, we can argue classically enough to obtain $2n \triangleright (\sigma, \tau)$, $2n{+}1 \triangleright (\sigma', \tau')$, and $B\,|\tau|\,b$. Then by Lemma 1 we obtain $\neg(R\,A\,|\tau|\,b)$, contradicting $R\,A = B$. □

Post's Theorem To connect the arithmetical hierarchy with the structure of Turing degrees, we again follow a usual textbook presentation translated to constructive type theory. To be able to state the theorem in our setting, we render all involved notions synthetically.

First, we represent the arithmetical hierarchy with a mutually inductive predicate:

$$\frac{f : \mathbb{N}^k \to \mathbb{B}}{\Sigma^k_0(\lambda \vec{n}.\, f\,\vec{n} = \mathsf{true})} \qquad \frac{f : \mathbb{N}^k \to \mathbb{B}}{\Pi^k_0(\lambda \vec{n}.\, f\,\vec{n} = \mathsf{true})} \qquad \frac{\Pi^{k+1}_n p}{\Sigma^k_{n+1}(\lambda \vec{n}.\, \exists x.\, p\,(x :: \vec{n}))} \qquad \frac{\Sigma^{k+1}_n p}{\Pi^k_{n+1}(\lambda \vec{n}.\, \forall x.\, p\,(x :: \vec{n}))}$$

The first two rules assert that k-ary decidable predicates form the base of the hierarchy. The third rule states that for a Π_n predicate p of arity $k{+}1$ the k-ary predicate obtained by capturing the first variable of p by an existential quantifier is Σ_{n+1}. The fourth rule dually expresses how a Σ_n predicate is turned into Π_{n+1} with a universal quantifier. As a sanity check, using a form of Church's thesis for a concrete model of computation, we can show the equivalence of our synthetic characterisation of the arithmetic hierarchy with a more conventional definition using first-order fomulas in the language of arithmetic, as mechanised in [5].

Secondly, we define the Turing jump A' of a predicate A using the core enumeration r_n:

$$A' := \lambda n.\, \exists R.\, (\forall f.\, R\,f = r_n\,f) \wedge R\,A\,n\,\mathsf{true}$$

This definition expresses the self-halting problem for oracle machines as it contains exactly those numbers n such that the n-th oracle machine R (as characterised by r_n) used with an oracle for A accepts n. We denote the n-th Turing jump of the empty predicate by $\emptyset^{(n)}$.

Finally, we say that A is semi-decidable relative to B if there is an oracle machine R with

$$\forall n.\, A\,n \leftrightarrow R\,B\,n\,\mathsf{true}.$$

The hardest part of Post's theorem is to show that $R\,A$ is Σ_1 relative to A by showing:

Lemma 2. *Given an oracle machine R with core r, termination $R\,A\,n\,b$ is equivalent to*

$$\exists L_{\mathsf{true}} L_{\mathsf{false}}.\, (\forall n \in L_{\mathsf{true}}.\, A\,b\,\mathsf{true}) \wedge (\forall n \in L_{\mathsf{false}}.\, A\,b\,\mathsf{false}) \wedge r\,(\mathsf{lookup}\,L_{\mathsf{true}}\,L_{\mathsf{false}})\,n = b$$

where $\mathsf{lookup}\,L_{\mathsf{true}}\,L_{\mathsf{false}}\,n$ returns true if $n \in L_{\mathsf{true}}$, false if $n \in L_{\mathsf{false}}$, and diverges otherwise.

We conclude Post's theorem in a common formulation, employing our synthetic definitions.

Theorem 2 (Post)**.** *Assuming LEM ($\forall p.\, p \vee \neg p$), the following can be shown:*

- *A unary predicate A is Σ_{n+1} iff it is semi-decidable relative to $\emptyset^{(n)}$.*
- *If A is Σ_n, then $A \preceq_T \emptyset^{(n)}$. If $n > 0$ already $A \preceq_m \emptyset^{(n)}$ for synthetic many-one reductions.*

In our current mechanisation, we assume LEM to allow switching between Σ_n and Π_n by complementation. We currently investigate how this assumption can be weakened or eliminated.

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